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Viscous heating of high Prandtl number fluids with temperature-dependent viscosity

E. R. G. ECKERT* and M. FAGHRI†

* University of Minnesota, Department of Mechanical Engineering, Minneapolis, MN 55455, U.S.A.
† University of Rhode Island, Department of Mechanical Engineering and Applied Mechanics, Kingston, RI 02881, U.S.A.

Abstract—The transient velocity and temperature fields have been calculated for the sudden start of Couette flow for fluids with Prandtl numbers of 1, 100 and 1000 and with a viscosity which decreases exponentially with temperature. The velocity field established itself much more rapidly than the temperature field for the fluids with Prandtl numbers 100 and 1000. The development of the temperature field occurs only after the velocity field is already established for fluids with Prandtl numbers larger than 1000 when the Nahme number has values smaller than 80.

1. INTRODUCTION

HEAT generation due to viscous dissipation in flowing fluids acts like a heat source. The temperature increase resulting in this way is especially large when either the velocity gradient or the viscosity of the fluid assumes large values. A large viscosity is also reflected in large values of the Prandtl number. For oils used in lubrication the Prandtl numbers are of the order of 10^2–10^4 whereas for liquid polymers they assume values up to 10^8. The viscosity of such liquids also decreases strongly with temperature. These characteristics create special features in the flow processes which are described extensively in the literature. Some representative papers are listed in refs. [1–8]. The temperature fields caused by viscous heating of such fluids were analyzed by Winter [5]. It was pointed out in ref. [9] that viscous heating of such fluids can lead to situations in which the fluid temperature increases exponentially in a process referred to as hydrodynamic thermal burst. References [10–12] also consider this process.

The present paper studies the velocity and temperature profiles generated under the influence of viscous heating in an unsteady starting process for fluids whose viscosity is exponentially dependent upon temperature. This problem is governed by two dimensionless groups: the Nahme number and the Prandtl number. It appears that such a situation has not been investigated before. A simple geometry has been selected to bring out clearly the basic features of such flow and heat transfer processes. The results are obtained for a range of Prandtl numbers and Nahme numbers.

2. FORMULATION

Conservation equations and boundary conditions

The geometric model selected for the study is shown schematically in Fig. 1. The fluid occupies an annular space of width b between two concentric cylinders. Initially, everything is at rest and at uniform temperature T_0. At time $\tau = 0$ the inner cylinder is suddenly set in motion and rotates with a constant angular velocity whereas the outer cylinder remains at rest. In the course of time, the fluid gradually participates in the motion and viscous dissipation increases the temperature. From rotational symmetry it follows that the velocity vector u points in the circumferential direction and depends only on the radial position. This is also true for the temperature T. The temperatures of the two boundaries of the fluid are maintained at the original value T_0 by cooling. The thickness of the fluid layer b is assumed to be small compared to the radius R so that the problem becomes identical to Couette flow with uniform pressure in the flow direction.

The momentum equation under these conditions for which the convective inertia terms disappear reduces to

$$\rho \frac{\partial u}{\partial \tau} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad (1)$$

where u denotes the local velocity at the distance y from the outer cylinder, ρ and μ are the density and the viscosity, respectively. The energy equation

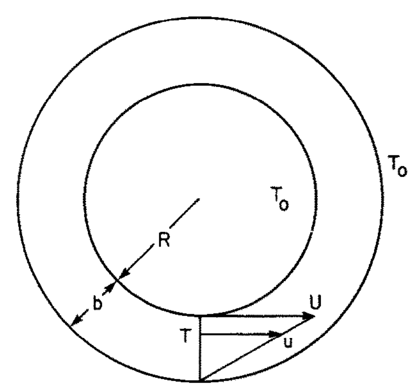

FIG. 1. Geometry considered in the present study. The thickness b of the annular fluid layer is assumed small compared with the radius R.

NOMENCLATURE

b width of the annular space in Fig. 1
c specific heat
k thermal conductivity
T temperature
t temperature measured with reference to T_0
U velocity of the inner cylinder in Fig. 1
u local velocity
y coordinate normal to the flow direction.

Greek symbols
β a parameter defined by equation (6)
ρ density
μ viscosity
τ time.

Dimensionless parameters
Na Nahme number defined by equation (9)
Pr Prandtl number defined by equation (9).

Indices
c refers to constant viscosity
0 refers to temperature T_0.

simplifies to

$$\rho c \frac{\partial T}{\partial \tau} = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \qquad (2)$$

with c denoting the specific heat and k the thermal conductivity.

The initial and the boundary conditions are:

$$\tau \leqq 0: \quad u = 0, \quad T = T_0 \qquad (3)$$

$$y = 0: \quad u = 0, \quad T = T_0 \qquad (4)$$

$$y = b: \quad u = U, \quad T = T_0. \qquad (5)$$

The velocity U of the inner cylinder is equal to $R\Omega$ where Ω is the angular velocity.

Properties

The density, specific heat and thermal conductivity are assumed constant. The viscosity is frequently approximated by the equation

$$\mu = \mu_0 \mathrm{e}^{-\beta(T-T_0)} \qquad (6)$$

which was originally suggested by Ludwig Prandtl. For polymers, the viscosity depends on the shear rate as well as temperature. Equation (2.15) in ref. [5], which approximates the viscosity for shear flow at constant density, transforms for the flow in Fig. 1 to the relation

$$\mu = \left(\frac{b}{U} \frac{\mathrm{d}u}{\mathrm{d}y} \right)^{(1/m)-1} \mu_0 \, \mathrm{e}^{-\beta(T-T_0)} \qquad (7)$$

in which the term U/b denotes a reference shear and μ_0 is the viscosity at the reference shear and at the temperature T_0. The pressure dependence which is usually small has been neglected. The term m assumes the value 1 [and equation (7) reduces to equation (6)] at low shear rates and a value between 2 and 5 at shear rates occurring in polymer processing. The simpler equation (6) is used in the present study.

Dimensionless equations

Equations (1)–(5) are now made dimensionless with the introduction of the following dimensionless variables

$$\underline{\tau} = \frac{k\tau}{\rho c b^2}, \quad \underline{y} = \frac{y}{b}, \quad \underline{t} = \beta(T - T_0), \qquad (8)$$

$$\underline{\mu} = \frac{\mu}{\mu_0}, \quad \underline{u} = \frac{u}{U}$$

and dimensionless parameters

$$Na_0 = \frac{\mu_0 \beta U^2}{k}, \quad Pr_0 = \frac{\mu_0 c}{k} \qquad (9)$$

where Na is referred to as the Nahme number. The momentum equation (1) takes on the form

$$\frac{\partial \underline{u}}{\partial \underline{\tau}} = (Pr_0) \frac{\partial}{\partial \underline{y}} \left(\mathrm{e}^{-\underline{t}} \frac{\partial \underline{u}}{\partial \underline{y}} \right) \qquad (10)$$

and the energy equation (2) reduces to

$$\frac{\partial \underline{t}}{\partial \underline{\tau}} = \frac{\partial^2 \underline{t}}{\partial \underline{t}^2} + (Na_0) \mathrm{e}^{-\underline{t}} \left(\frac{\partial \underline{u}}{\partial \underline{y}} \right)^2. \qquad (11)$$

The dimensionless initial and boundary conditions are:

$$\underline{\tau} \leqq 0: \quad \underline{u} = 0, \quad \underline{t} = 0 \qquad (12)$$

$$\underline{y} = 0: \quad \underline{u} = 0, \quad \underline{t} = 0 \qquad (13)$$

$$\underline{y} = 1: \quad \underline{u} = 1, \quad \underline{t} = 0. \qquad (14)$$

The solution for the dimensionless temperature field and the dimensionless velocity field will have the form

$$(\underline{t} \text{ or } \underline{u}) = f(Na_0, Pr_0, \underline{\tau}, \underline{y}). \qquad (15)$$

For steady state, which will be approached asymptotically at large values of time, the left hand term in equation (1) is zero and the Prandtl number drops out of equation (10). The dimensionless temperature and velocity fields are described by equations of the form:

$$(\underline{t}\,\text{or}\,\underline{u}) = f(Na_0, \underline{\tau}, \underline{y}). \tag{16}$$

The asymptotic solutions are, therefore, independent of Prandtl number. The energy equation becomes:

$$\frac{d^2\underline{t}}{d\underline{y}^2} + (Na_0)\,e^{\underline{t}} = 0. \tag{17}$$

The maximum temperature $\underline{t}_{max}$ in the fluid is obtained by a solution of this equation. For constant viscosity it assumes the form

$$\underline{t}_{c,max} = \frac{Na_0}{8}. \tag{18}$$

Solution methodology

The numerical methodology and the presentation of the results will be described in this section. The primary tool used in solving the coupled partial differential equations (10) and (11) and their associated initial and boundary conditions is the Patankar–Spalding method [13]. This is a fully implicit, finite-difference scheme designed for two-dimensional parabolic problems. The non-linear viscous dissipation term on the RHS of equation (10) was linearized and was introduced as an extra source term in the program. The solution is obtained by starting with known values at $\underline{\tau} = 0$ and marching with time in the direction of increasing $\underline{\tau}$. Multiple iteration was required at each time step as a result of the non-linearity and inter-linkage in the governing equations.

The solution was carried out with 100 grid points in the region $0 \leqslant y \leqslant 1$ and, in the $\underline{\tau}$ direction, the computation was continued with time steps as small as 10^{-6} until asymptotic values for velocity and temperature were reached for most cases.

Aside from the accuracy tests involved with the step size studies, comparison of the results were made with the asymptotic steady-state solution by Nahme [1]. The results agreed to within 0.1%.

Representative results describing the velocity and temperature field will now be described.

3. RESULTS

Figures 2–4 present profiles of the dimensionless velocity $\underline{u}$ and of the dimensionless temperature $\underline{t}$ as a function of the dimensionless distance $\underline{y}$. The Prandtl number varies in the figures in a vertical direction having the values 1, 100 and 1000. The Nahme number has the value 8 in Fig. 2 and the value 80 in Fig. 3.

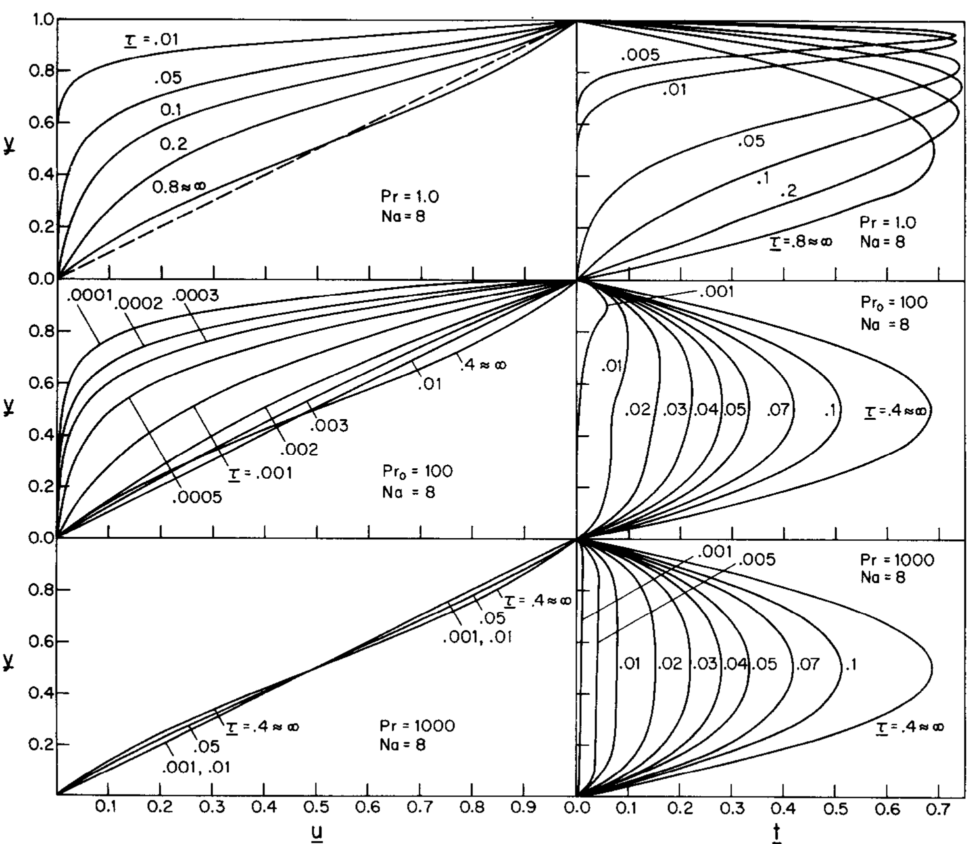

FIG. 2. Dimensionless velocity profiles $\underline{u}$ and temperature profiles $\underline{t}$ for a Nahme number 8 and for three Prandtl numbers. The boundaries of the fluid layer are kept at the dimensionless temperature $\underline{t} = 0$.

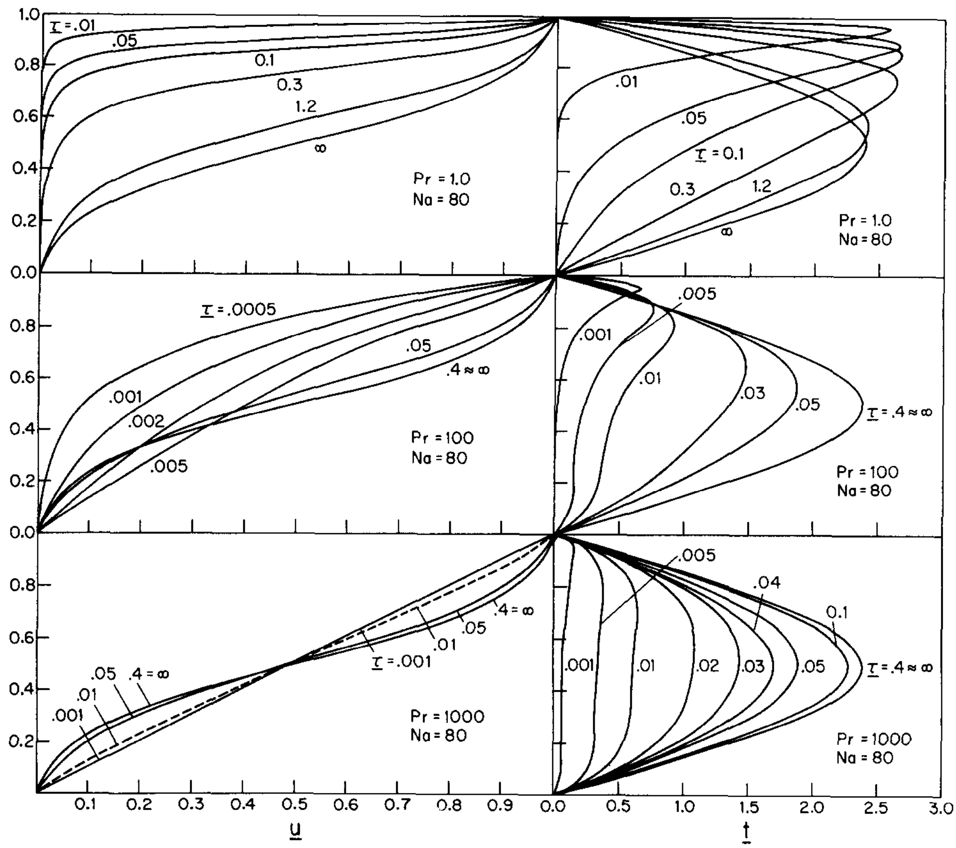

FIG. 3. Dimensionless velocity profiles $\underline{u}$ and temperature profiles $\underline{t}$ for a Nahme number of 80 and for three Prandtl numbers. The dimensionless temperature $\underline{t}$ at the boundaries is maintained at a value 0.

Figure 4 contains a selected number of diagrams with various Prandtl and Nahme numbers. The dimensionless time $\underline{\tau}$ is the parameter on the curves.

Steady state

The asymptotic steady state will be discussed at first. It can be observed in Figs. 2 and 3 that the asymptotic velocity and temperature profiles are identical for the three Prandtl numbers, confirming equation (16). The velocity profiles are S-shaped. This can be readily understood considering the fact that the shear stress σ is, in the steady state, constant throughout the layer. According to the equation $\tau = \mu(du/dy)$, the velocity gradient is large in the center part of the layer where the temperature is high and the viscosity is small and the gradient is smaller near the two walls where the temperature is lower and the viscosity larger. For a temperature-independent viscosity the velocity profile is a straight line as shown by the dashed curve in Fig. 2.

Figure 5 compares the actual velocity profiles u of a fluid with constant viscosity with the profile of a fluid with a viscosity decreasing with increasing temperature. The velocity U at the border is kept constant in the comparison for the left-hand figure and the wall shear (and therefore the velocity gradient) is kept constant for the right-hand figure. It is obvious that the wall shear is smaller for the variable viscosity fluid than for the constant property one where the velocity U is constant, but that the velocity U is larger for the variable viscosity fluid than for the constant property fluid when the wall shear σ_0 is constant in the comparison. It will be shown later that this is the root of the hydrodynamic thermal burst.

For the condition σ_0 = const. the Nahme number has to be defined using the prescribed wall shear instead of the velocity. It then has the form

$$Na = \frac{\beta \sigma_0^2 b^2}{\mu_0 k}. \tag{19}$$

The effect of viscous heating is very pronounced in engineering developments which operate with gas or air streams at high subsonic or supersonic velocities, for instance in gas turbines, high speed aeroplanes and space vehicles. The effect is there called 'aerodynamic heating' and the Eckert number

$$Ec = \frac{U^2}{c\Delta T} \tag{20}$$

has occasionally been used to describe it quantitatively [14].

Gases have Prandtl numbers of order 1. The dia-

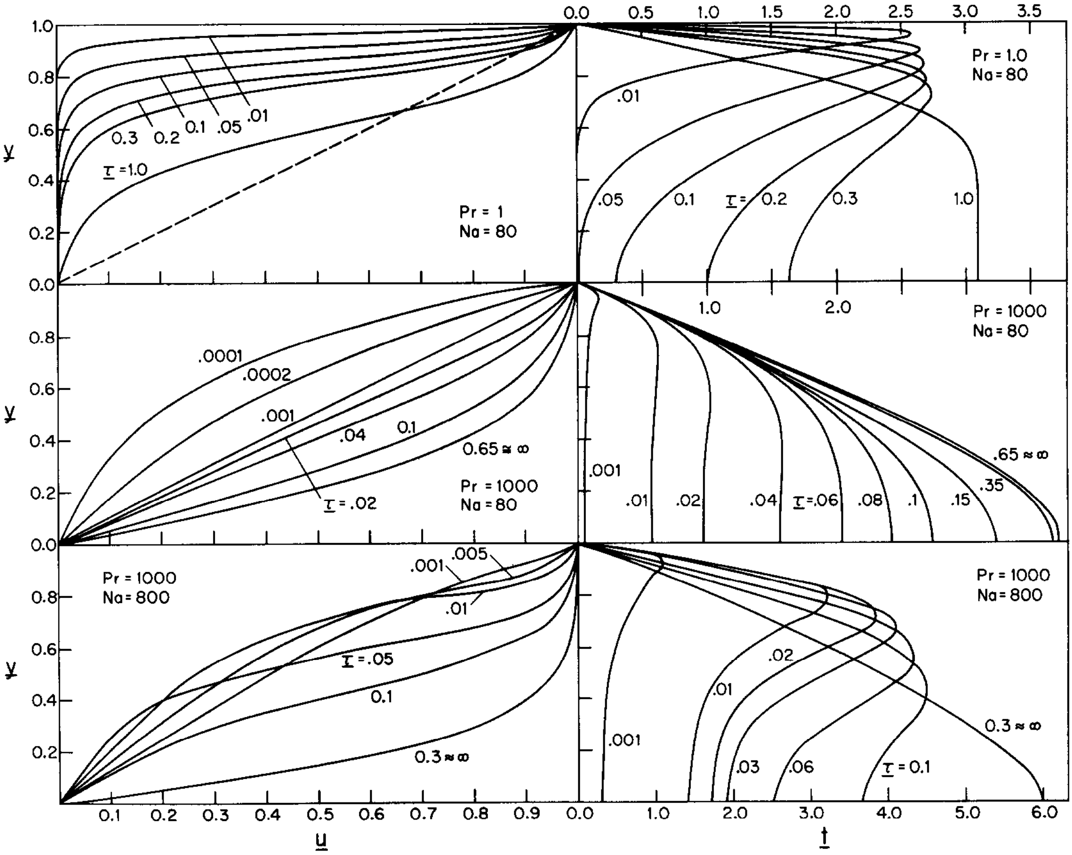

FIG. 4. Dimensionless velocity profiles $\underline{u}$ and temperature profiles $\underline{t}$ for three combinations of the Nahme and Prandtl numbers. The temperature $\underline{t}$ at $\underline{y} = 0$ is maintained at zero and the boundary at $\underline{y} = 1$ is adiabatic.

grams for this Prandtl number, in Figs. 2–4, however, do not describe aerodynamic heating quantitatively because the temperature dependence of gases is not described by equation (6). The trends in their behavior should be represented by the figures. The parameter in equation (20) takes on the form:

$$Ec_\beta = \frac{\beta U^2}{c} \tag{21}$$

when $1/\beta$ is used as the prescribed term with the dimension of a temperature. The Nahme number is then the product of Eckert and Prandtl numbers:

$$Na = Ec_\beta Pr. \tag{22}$$

The fact that the temperature and velocity fields depend for the present situation only on the product $Ec_c Pr$ for large Prandtl numbers is analogous to the situation that the Nusselt number depends in many forced-convection problems only on the product of Reynolds and Prandtl number, the Stanton number St

$$St = RePr \tag{23}$$

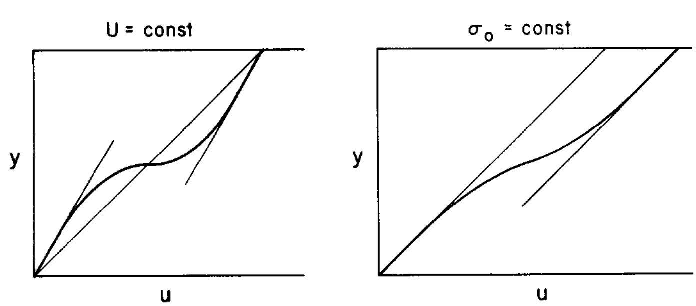

FIG. 5. Velocity profiles for the boundary condition $U = 0$ and $\sigma_0 = 0$.

or in natural convection only on the product of Grashof number Gr and of Prandtl number. The dimensionless Stanton number St and the Rayleigh number Ra have been defined accordingly as shown in equations (23) and (24).

$$Ra = GrPr \tag{24}$$

Transient state

The transient temperature profiles in Figs. 2 and 3 show that the time $\underline{\tau}$ required for the development of the temperature profile does not depend strongly on Prandtl number. It is somewhat larger at $Pr = 1$ than at $Pr = 100$ or 1000. The situation is radically different for the velocity profile. For a Prandtl number of 1, the time to develop the steady velocity profile is of the same order of magnitude as the time required for the development of the temperature profile. At a Prandtl number of 100, the velocity profile develops much faster than the temperature profile and for a Prandtl number of 1000, the velocity profile has already developed before the temperature in the fluid layer has increased to any amount. Many papers concerned with the viscous heating effect of large Prandtl number fluids are, therefore, correctly based on the assumption that the temperature profile starts to develop only after the velocity profile has already acquired its steady-state value and that the left-hand term in the momentum equation (1) can be approximated by the value zero. Physically this means that the effect of inertia has vanished and that shear is the only remaining force. The shear stresses are then uniform in magnitude across the fluid layer for the Couette flow situation.

The time requiıed to reach asymptotically the steady state can be divided into two parts for fluids with a Prandtl number larger than 1000 and for a Nahme number smaller than 80. In the first part, occurring immediately after the start of the flow, the effect of inertia is still strong and influences the development of the velocity profile whereas the temperature is practically maintained at the original value. In the second part of the development time, the effect of the inertia has vanished and the shear is distributed uniformly throughout the fluid. The velocity profile changes during this time period somewhat from a straight line to an S-shaped curve under the influence of the varying viscosity. The temperature in the fluid increases and reaches asymptotically a steady state for which, in any cross-section, all of the energy dissipated into heat is conducted away to the boundaries of the fluid layer. In this part of the time τ the velocity and temperature fields and the heat transfer to the walls do not depend on Prandtl number but only on Nahme number. The first part of the development time is very short compared to the second part so that the average heat transfer is also a function of Nahme number alone.

During the second part of the development, the velocity profile changes its shape from the straight line to an S-shaped curve as shown on the left-hand side of Fig. 6 where a constant velocity U is maintained and as shown on the right-hand side of the figure where a constant shear stress σ_0 is maintained. A runaway condition called 'hydrodynamic thermal burst' in which the velocity U and the temperature in the fluid increase exponentially may occur at a large Nahme number where a constant shear (or a constant torque for the cylindrical configuration) is maintained.

REFERENCES

1. R. Nahme, Beiträge zur hydrodynamischen Theorie der Lagerreibung, *Ing.-Archiv* **11**, 191 (1940).
2. D. D. Joseph, Variable viscosity effects on the flow and stability of flow in channels and pipes, *Phys. Fluids* **7**, 1761–1771 (1964).
3. D. D. Joseph, Stability of frictional-heated flow, *Phys. Fluids* **8**, 2195–2200 (1965).
4. D. D. Joseph, Boundary conditions for thin lubrication layers, *Phys. Fluids* **23**, 2356–2358 (1980).
5. H. H. Winter, Viscous dissipation in shear flows of molten polymers, *Advances in Heat Transfer*, pp. 205–267. Academic Press, New York (1977).
6. J. R. A. Pearson, Variable-viscosity flows in channels with high heat generation, *J. Fluid Mech.* **83**, 191–206 (1977).
7. J. R. A. Pearson, Polymer flows dominated by high heat generation and low heat transfer, *Polymer Engng Sci.* **18**, 222–229 (1978).
8. h. ockendon, channel flow with temperature-dependent viscosity and internal viscous dissipation. *J. Fluid Mech.* **93**, 737–746 (1979).
9. S. A. Bostandzhiyan and A. M. Stolin, The critical conditions of the thermal regime of generalized Couette flow, *Inzh.-Fiz. Zh.* **17**, 86–94 (1969).
10. A. M. Stolin, S. A. Bostandzhiyan and N. V. Plotnikova, Conditions for occurrence of hydrodynamic thermal explosion in flows of power-law fluids, *Heat Transfer—Soviet Res.* **10**, 86–93 (1978).
11. R. O. Ayeni, On the thermal runaway of variable viscosity flows between concentric cylinders, *Z. angew. Math. Mech.* **23**, 408–413 (1982).
12. Y. A. Buevich and M. I. Zaslvaskii, Hydrodynamic thermal burst in radial bearing, *J. Engng Phys.* **42**, 566–571 (1982).
13. S. V. Patankar and D. B. Spalding, *Heat and Mass Transfer in Boundary Layers*, 2nd edn. Intertext, London (1970).
14. E. R. G. Eckert and R. M. Drake, Jr., *Analysis of Heat and Mass Transfer*, Chap. 10. McGraw-Hill, New York (1972).

CHAUFFAGE VISQUEUX DE FLUIDES A GRANDS NOMBRES DE PRANDTL AVEC UNE VISCOSITE DEPENDANT DE LA TEMPERATURE

Résumé—Les champs variables de température et de vitesse sont calculés pour le démarrage brutal de l'écoulement de Couette de fluides ayant des nombres de Prandtl de 1, 100 et 1000 et une viscosité qui décroît exponentiellement avec la température. Le champ de vitesse s'établit plus vite que le champ de température pour les nombres de Prandtl 100 et 1000. Le développement du champ de température n'est atteint qu'une fois le champ de vitesse stabilisé pour les nombres de Prandtl supérieurs à 1000 lorsque le nombre de Nahme a des valeurs inférieures à 80.

VISKOSEBEHEIZUNG VON FLUIDEN MIT HOHER PRANDTL-ZAHL UND TEMPERATURABHÄNGIGER VISKOSITÄT

Zusammenfassung—Es wurde das transiente Geschwindigkeits- und Temperaturprofil für den plötzlichen Anlauf einer Couette-Strömung für Fluide mit Prandtl-Zahlen von 1, 100, 1000 und einer mit der Temperatur exponentiell abnehmenden Viskosität berechnet. Das Geschwindigkeitsprofil bildet sich bei Fluiden mit Prandtl-Zahlen von 100 und 1000 sehr viel schneller aus, als das Temperaturprofil. Bei Fluiden mit Prandtl-Zahlen von mehr als 1000 erfolgt die Ausbildung des Temperaturprofils nur dann langsamer als die Ausbildung des Geschwindigkeitsprofils, wenn die Nahme-Zahl kleiner als 80 ist.

ВЯЗКОСТНЫЙ НАГРЕВ ЖИДКОСТЕЙ С БОЛЬШИМ ЧИСЛОМ ПРАНДТЛЯ И ВЯЗКОСТЬЮ, ЗАВИСЯЩЕЙ ОТ ТЕМПЕРАТУРЫ

Аннотация—Нестационарные поля. скорости и температуры рассчитаны для внезапного начала течения Куэтта для жидкостей с числами Прандтля 1, 100 и 1000 и вязкостью, уменьшающейся по экспоненциальному закону с уменьшением температуры. Для жидкостей с числами Прандтля 100 и 1000 поле скорости устанавливается значительно быстрее, чем поле температуры. Для жидкостей с числом Прандтля более 1000 и числах Наме меньших 80, развитие поля температуры происходит только после установления поля скорости.